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OSCILLATIONS OF ONE-DIMENSIONAL LINEAR SYSTEMS WITH MOBILE WEIGHTS UNDER THE ACTION OF LONGITUDINAL IMPACTS

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L. A. MANASHKIN
(Dnepropetrovsk)

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Longitudinal osciallations of a one-dimensional system which can be represented by a rod interacting with various kinds of inertial mobile media, are considered. It is assumed that the media do not react with each other, can only move along the rod and, that there is no internal interaction between the elements of the media. The model can be used to study the oscillations of sufficiently long chains of rigid bodies to which other mobile bodies are attached by means of deformable elements, oscillations of one-dimensional systems of rigid bodies with cavities partially filled with fluid, etc. A transitional mode of motion in similar systems was studied in [1].

1. Let us consider a rod of length l_1 colliding at initial velocity v_0 with an identical stationary rod of length l_2. The collision produces a new rod of length $l = l_1 + l_2$. The motion of such a system is described by the following differential equations with initial and boundary conditions:

$$\rho_1 \frac{\partial^2 u}{\partial t^2} + \sum_{j=1}^{n} P_j = \frac{\partial S}{\partial x} \tag{1.1}$$

$$\begin{gathered}
\rho_{2j}\partial^2 w_j / \partial t^2 + P_j = \rho_{2j}\partial^2 u / \partial t^2, \; j = 1,2, \ldots, n \\
\partial u(0, t) / \partial x = \partial u(l, t) / \partial x = 0, \; u(x, 0) = w_j(x, 0) = 0 \\
\partial w_j(x, 0) / \partial t = 0, \; j = 1,2, \ldots, n, \; \partial u(x, 0) / \partial t = v_0 [1 - \sigma_0(x - l_1)]
\end{gathered} \tag{1.2}$$

Here ρ_1 is the density of the rod, ρ_{2j} is the density of the j-th medium; $u(x, t)$ denote the longitudinal displacements at the instant t of the cross section of the rod at the distance x from one of its free ends (the other free end has the coordinate $x = l$); $w_j(x, t)$ is the displacement of the j-th medium relative to the rod; S is the longitudinal force acting at the cross section of the rod; P_j is the intensity of the force of interaction of the rod with the j-th medium and $\sigma_0(x - l_1)$ denotes the unit Heaviside function.

Let us write Eq. (1.1) in the form

$$\rho \mathbf{J}_r \left[\frac{\partial^2 u}{\partial t^2}\right] = \frac{\partial S}{\partial x}, \quad \rho = \rho_1 + \sum_{j=1}^{n} \rho_{2j} \tag{1.3}$$

$$\mathbf{J}_r[z] = [z] - \sum_{j=1}^{n} r_{2j} \frac{\partial^2}{\partial t^2} \int_0^t G_j(t-\tau)[z]\, d\tau, \quad r_{2j} = \frac{\rho_{2j}}{\rho}$$

Here $G_j(t-\tau)$ is the Green's function defining the forces of interaction with the media. Seeking the solution of (1.3) in the form $u(x,t)=X(x)\,T(t)$, we obtain

$$X''+(\lambda/l)^2X=0,\quad \rho J_r[T^{\cdot\cdot}]+(\lambda/l)^2\,S[T]=0 \tag{1.4}$$

Here S is an operator (see below), λ is an eigenvalue which, under the conditions (1.2) is equal to $0, \pi, 2\pi, \ldots, i\pi, \ldots$, and $X_i(x)=\cos(\lambda x/l)$ are the corresponding eigenfunctions.

We shall seek the solution in the form

$$u(x,t)=X_0q_0(t)+\sum_{i=1}^{\infty}X_i(x)\,q_i(t) \tag{1.5}$$

$$w_j(x,t)=X_0\varphi_{0j}(t)+\sum_{i=1}^{\infty}X_i(x)\,\varphi_{ij}(t)$$

Here $q_0,\ldots,q_i,\ldots,\varphi_{0j},\ldots,\varphi_{ij}$ denote the generalized coordinates [2] given by the differential equations

$$a_{00}{}^iq_i^{\cdot\cdot}+\sum_{j=1}^{n}a_{0j}{}^i\varphi_{ij}^{\cdot\cdot}+z_{1i}S[q_i]=0 \tag{1.6}$$

$$a_{j0}^iq_i^{\cdot\cdot}+a_{jj}{}^i\,\varphi_{ij}^{\cdot\cdot}+z_{2i}\mathbf{P}_j[\varphi_{ij}]=0,\quad i=0,1,2,\ldots;\quad j=1,2,\ldots,n$$

$$a_{00}{}^i=\rho z_{2i},\quad a_{j0}{}^i=a_{0j}{}^i=-r_{2j}a_{00}{}^i,\quad a_{jj}{}^i=r_{2j}a_{00}{}^i$$

$$z_{1i}=\int_0^l(X_i')^2\,dx,\quad z_{2i}=\int_0^lX_i^2dx$$

where S and $\mathbf{P}_j$ are operators the structure of which determines the dependence of the magnitudes of the forces S and $\mathbf{P}_j$ on the deformation $\partial u/\partial x$ and displacements $w_j(x,t)$.

In accordance with (1.2) we obtain, for (1.6),

$$q_i(0)=\varphi_{ij}(0)=0,\quad q_i^{\cdot}(0)=v_0\frac{z_{3i}}{z_{2i}},\quad \varphi_{ij}(0)=0,\quad z_{3i}=\int_0^{l_1}X_i^2dx \tag{1.7}$$

Applying the Laplace transformation to (1.6) with conditions (1.7), we obtain the following system of algebraic equations:

$$a_{00}{}^ip^2q_i(p)+\sum_{j=1}^{n}a_{0j}{}^ip^2\varphi_{ij}(p)+z_{1i}S[p]\,q_i(p)=\Phi_i \tag{1.8}$$

$$a_{j0}{}^ip^2q_i(p)+a_{jj}{}^ip^2\varphi_{ij}(p)+z_{2i}P_j[p]\,\varphi_{ij}(p)=-r_{2j}\Phi_i$$

$$i=0,1,2,\ldots;\ j=1,2,\ldots,n$$

where p is a complex variable, $q_i(p)$, $\varphi_{ij}(p)$, $S[p]$, $P_j[p]$ are the transforms of the variables and $\Phi_i=\rho\,v_0z_{3i}$.

Solutions of (1.8) are given by the expressions

$$q_i(p)=D_i{}^q(p)/D_i(p),\ \varphi_{ij}(p)=D_{ij}{}^\varphi(p)/D_i(p) \tag{1.9}$$

$$D_i = \{T^i - \sum_{j=1}^{n} a_{0j}{}^i p^2 R_j{}^i\} \prod_{j=1}^{n} Q_j^i$$

$$D_i{}^q = \Phi_i \{1 + \sum_{j=1}^{n} r_{2j} R_j{}^i\} \prod_{j=1}^{n} Q_j^i$$

$$D_{ij}{}^\varphi = - \Phi_i \{r_{2j} [T^i - \sum_{=1}^{n} a_{0k}{}^i p^2 (1 - \delta_{kj}) R^i{}_k] +$$

$$a_{0j}{}^i p^2 [1 + \sum_{k=1}^{n} r_{2k} (1 - \delta_{kj}) R^i{}_k]\} \prod_{k=1}^{n} (Q_k{}^i)^{1-\delta_{kj}}$$

$$Q_j{}^i = a_{jj}{}^i p^2 + z_{2i} P_j [p], \quad R_j{}^i = a_{0j}{}^i p^2 (Q_j{}^i)^{-1}, \quad T^i = a_{00}{}^i p^2 + z_{1i} S [p]$$

where δ_{kj} is the Kronecker delta.

The transforms of the functions describing the variation of the forces S with time and of the accelerations a_j of the media with indices $j = 1, 2, \ldots, n$, have the form

$$S(p) = \sum_{i=1}^{\infty} X_i{}'(x) S[p] q_i(p) \tag{1.10}$$

$$a_j = (r_{2j} p)^{-1} \sum_{i=0}^{\infty} X_i(x) P_j [p] \varphi_{ij}(p)$$

From (1.9) and (1.10) we obtain the following expresssions for the case in which the equations $D_i(p) = 0$ have complex roots (h_{ij} and ω_{ij} denote the real and imaginary parts of the pair of roots with index j):

$$S(\xi, t) = - v_0 \sum_{i=1}^{\infty} f(i) \sin i\pi\xi_0 \sin i\pi\xi \sum_{j=1}^{n+1} V_{ij}, \quad \xi = \frac{x}{l} \tag{1.11}$$

$$a_j(\xi, t) = - v_0 \sum_{i=1}^{\infty} if(i) \sin i\pi\xi_0 \cos i\pi\xi \sum_{j=1}^{n+1} W_{ij}, \quad \xi_0 = \frac{l_1}{l}$$

$$V_{ij} = A_{ij} \exp(-h_{ij}t) \cos(\omega_{ij}t + \gamma_{ij}), \quad W_{ij} = B_{ij} \exp(-h_{ij}t) \cos(\omega_{ij}t + \varepsilon_{ij})$$

For each pair of real roots of a single equation, the functions V_{ij} and W_{ij} are obtained in the form

$$V_{ij} = A_{ij} \exp(-h_{ij}t) + \gamma_{ij} \exp(-\omega_{ij}t) \tag{1.12}$$

$$W_{ij} = B_{ij} \exp(-h_{ij}t) + \varepsilon_{ij} \exp(-\omega_{ij}t)$$

The quantities $A_{ij}, B_{ij}, \gamma_{ij}$ and ε_{ij} in (1.11) and (1.12) are functions of the initial parameters of the system, of the eigenvalues λ_i and of the corresponding values of h_{ij} and ω_{ij}, and $f(i)$ is the convergence multiplier used for smoothing the spikes caused by the Gibbs phenomenon.

Thus the solution (1.11) represents a sum of packets of particular solutions with the same distribution of amplitudes along the length of the system. The packet contains $n + 1$ components where n is the number of mobile media with which the rod interacts.

2. Let us consider the longitudinal forces S and accelerations a_j for the simplest case of $n = 1$. This case corresponds, e. g., to the longitudinal oscillations of a train of carriages with mobile amortized loads or of a train of cisterns, if the motion of the liquid within them is represented by a discrete mechanical analog and the higher oscillation frequencies of the liquid in the cisterns, which have little effect on the motion of the system, are neglected [4].

Let us adopt $S = k(1 + \mu\partial / \partial t)$, and $P_1 = k_1(1 + \mu_1 \partial / \partial t)$. We shall assume that the coefficients μ and μ_1 are constants of viscoelastic systems and for the elastic systems with hysteresis [2] we have $\mu = \mu_i = \mu' / \nu_{i1}$, $\mu_1 = \mu_{1i} = \mu_1' / \nu_{i2}$ (ν_{i1} and ν_{i2} are the eigenfrequencies of a conservative system, and μ', μ_1' are dimensionless coefficients characterizing the dissipation of energy when the phase angle is changed by one radian).

In the case of $n = 1$ for $\mu = \mu_1 = 0$, the equations $D_i(p) = 0$ correspond to the natural oscillations of conservative systems with two degrees of freedom. Consequently, for each eigenvalue λ_i of the system, we have a corresponding packet of two characteristic oscillations with the same distribution of amplitudes. On increasing the values of λ_i, the frequencies of these oscillations tend asymptotically to the values $\nu_{i1}^* = \lambda_i [k / \rho(1 - r_{21})]^{1/2} / l$ of the frequencies of oscillation of a rod of length l, and to the value $\nu_2^* = [k_1 / \rho r_{21}]^{1/2}$ of the partial oscillation frequency of the medium with respect to the fixed rod.

As an example, we determine the forces and accelerations appearing when two equal trains, each containing 10 railway carriages, each carriage of 13. 9m length, collide at the velocity of 1 m/s, in the case when there are no gaps in the intercarriage links. We assume that $\rho = 6.1$ ton/m, $k = 320$ mN and $r_{21} = 0.5$. The value of k_1 varies within the range defined by the relation $0.005 \leqslant \alpha \leqslant 1$ where $\alpha = 13.9^2 k_1 / k$.

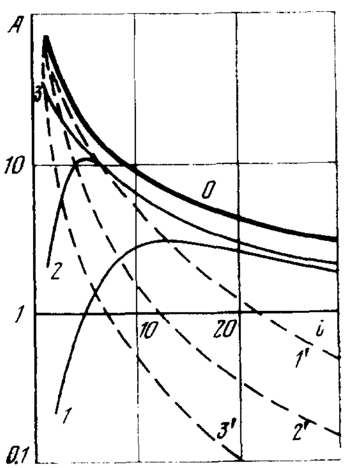

Fig. 1

Figure 1 depicts the change in the value of the coefficients A_{i1} (curves *1, 2, 3*) and A_{i2} (curves *1', 2', 3'*) with i, in the relations (1. 11) when $\mu = \mu_1 = 0$. The curves *1* and *1'* correspond to $\alpha = 1$, *2* and *2'* to $\alpha = 0.1$, and *3*, *3'* to $\alpha = 0.01$. Curve *0* depicts the variation in the value of the coefficients A_i in the case of a solid rod ($r_{21} = 0$). from Fig. 1 we see that

$$A_{i1} + A_{i2} \leqslant A_i \qquad (2.1)$$

in all cases. The coefficients A_{i1} tend with increasing i to the corresponding values of A_i. The uniform convergence of the series (1. 11) with the smoothing multipliers [3] when $r_{21} = 0$, and the condition (2. 1), together imply that when $r_{21} \neq 0$ [5], then the series (1. 11) converge to uniform, almost periodic functions.

Figure 2 shows the variation of the forces with time at various cross sections of one of the colliding systems. Curves *1* correspond to

$\xi = 0.5$ (at the site of contact of the colliding systems) and the values of 1/2 and 1/4 denote the lines constructed for $\xi = 1/4$ and 1/8 , respectively. Thick solid lines correspond to $\mu = \mu_1 = 0$. Thin solid lines correspond to the case of a viscoelastic system, and the broken lines to an elastic system with hysteresis (in the latter case $\mu' = \mu_1' = 0.1$). To make possible the comparison of the results obtained for the viscoelastic systems with those obtained for the elastic systems with hysteresis, the coefficients μ and μ_1 were determined using the relations $\mu = \mu' / \nu_{11}^*$, $\mu_1 = \mu_1' / \nu_{12}^*$ where ν_{11}^* is the frequency of the first harmonic of the partial oscillations of a load-free rod, and ν_{12}^* is the frequency of oscillations of the load relative to the stationary rod. Fig. 2a corresponds to the case $r_{21} = 0$ (oscillations of a solid rod

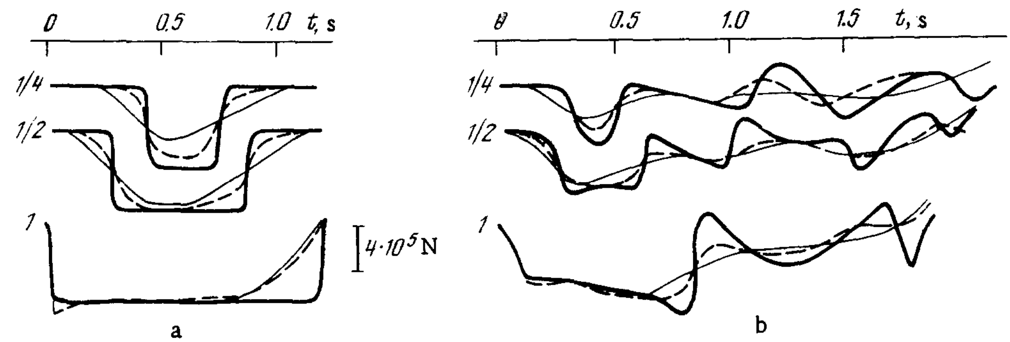

Fig. 2

during the collision) and Fig. 2b corresponds to the case $r_{21} = 0.5$, $\alpha = 0.01$. Analysing the above relations we find, that if a part of the mass of the one-dimensional system can move with respect to the rod, while retaining the elastic or viscoelastic coupling with the rod, then although the duration of intense activity of the forces is reduced, the time during which the compression is predominant is increased and determined by the time of propagation of the waves along the rod of density ρ_1 and ridigity k.

Figure 3 shows the dependence of the maximum values of the forces and accelerations (in units of g) on α. Curve *1* corresponds to a conservative system ($\mu' = \mu_1' = 0$), curve *2* to a viscoelastic and *3* to the elastic system with hysteresis. From Fig. 3 it is apparent that the maximum forces occurring within the interval of values of α may exceed the force arising in collisions of solid rods, by about 1.3 times. When the value of α is reduced from 1 to 10^{-2}, the acceleration of the loads is also diminished. The points *1'*, *2'* and *3'* in Fig. 3 denote the largest values of the forces during the collisions of the systems with

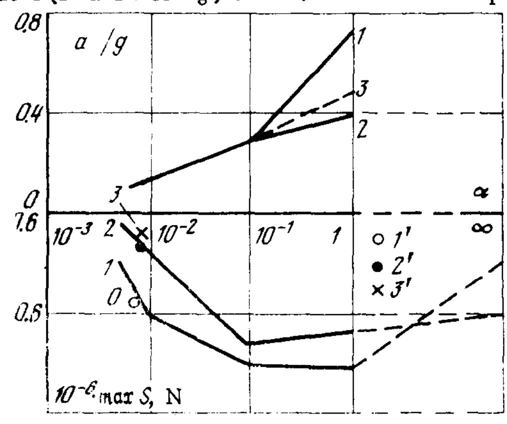

Fig. 3

cavities partially filled with fluid [4]. In these cases $r_{21} = 0.41$ and the point $1'$ corresponds to a conservative system, $2'$ to a viscoelastic and $3'$ to an elastic system with hysteresis. Analysis of the results deplicted in Fig. 2 and 3 shows that viscous dissipation of energy leads to increase in the values of the forces during the collision of solid rods, and to reduction when the colliding systems have mobile loads. The differences in the values of the forces and accelerations in the viscoelastic and elastic systems with hysteresis [1] are not essential.

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